

# Converting Zitterbewegung Oscillation to Directed Motion

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Zitterbewegung oscillation (ZB), namely, the jittering center-of-mass motion predicted by free-space Dirac (or Dirac-like) equations, has been studied in several different contexts. It is shown here that ZB can be converted to directed center-of-mass motion by a modulation of the Dirac-like equation, if the modulation is on resonance with the ZB frequency. Tailored modulation may also stop, re-launch or even reverse the directed motion of a wavepacket with negligible distortion. The predictions may be examined by current ZB experiments using trapped-ion systems.

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## I. INTRODUCTION

Zitterbewegung motion (ZB) originally refers to the trembling motion of a free relativistic particle (e.g., electron) as described by the Dirac equation [1]. ZB in this context has an extremely large frequency and a very small amplitude, so it is hardly observable. Nevertheless, the physics of ZB becomes highly relevant to a number of interesting situations where the quantum dynamics is governed by artificial Dirac or Dirac-like equations. Examples include band electrons in graphene [2], cavity electrodynamics [3], ultracold atoms in designed laser fields [4–8], macroscopic sonic crystals and photonic superlattices [9], as well as trapped-ion systems [10, 11]. For a review of recent fruitful studies of ZB physics using electrons in semiconductors, see Ref. [12].

Many quantum dynamical phenomena induced by a control field have been explored using the Schrodinger equation. Much less is known in the case of driven Dirac or Dirac-like equations. This work is concerned with ZB in a simple version of a driven Dirac-like equation. Previously, we showed that ZB in the cold-atom context as a quantum coherence phenomenon can be coupled with some mechanical oscillations, resulting in the control of the amplitude, the frequency, and the damping of the ZB associated with a wavepacket [8]. Going one step further, it should be of interest to study what happens to ZB if one term in a Dirac-like equation is modulated with a frequency on-resonance with the ZB frequency. Indeed, in both classical mechanics and quantum mechanics, it is always expected that an oscillation phenomenon can behave remarkably different when it is subject to an on-resonance driving field. A latest example is Bloch oscillation subject to on-resonance driving, where Bloch oscil-

lation may be converted to “super-Bloch oscillation” [13] or even to directed motion [14].

In this work we start from a Dirac-like equation that is already experimentally realized in the context of trapped-ion systems [11]. We then introduce a modulation to the Dirac-like equation, whose frequency is on resonance with the ZB frequency. Due to the on-resonance driving, it is shown, both analytically and numerically, that ZB associated with a wavepacket can be converted to directed wavepacket motion. By tailoring the modulation, we may also stop, re-launch or even reverse the directed motion of a wavepacket with negligible distortion. These results indicate that ZB is not a mysterious oscillation, and it provides an extra useful control knob for manipulating quantum motion.

## II. THEORETICAL ANALYSIS

Let us consider a simple version of a two-component-spinor Dirac-like equation (taking  $\hbar = 1$  throughout),

$$i \frac{d}{dt} |\psi\rangle = H |\psi\rangle = \frac{1}{2} (C \sigma_y + D p_x \sigma_x) |\psi\rangle, \quad (1)$$

where  $p_x$  is the momentum in the  $x$  direction;  $C$  and  $D$  are two c-number coefficients,  $\sigma_x, \sigma_y$  are Pauli matrices in the  $x$  and  $y$  directions. We first assume  $C$  is time-independent, with  $C = C_0 > 0$ . Time-dependence of  $C$  will be discussed much later. We also note that the Hamiltonian  $H$  here depends on  $p_x$ , but not on the momentum along the  $y$  direction. Such kind of Hamiltonian may be synthesized by tripod-scheme cold atoms with a strong constraint potential in the transverse direction [6], tripod-scheme cold atoms in a two-dimensional laser setup with mirror oscillations [8], and trapped-ion systems [10, 11]. Because the  $\sigma_x p_x$  term is the only spin-orbit coupling term, the ZB oscillation associated with a wavepacket does not suffer from a fast damping in its oscillation amplitude [6, 8].

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We now examine the motion of an initial state ( $t = 0$ ) as a product state of a one-dimensional wavepacket in  $x$  and an internal two-component spinor state,

$$\langle x|\psi(0)\rangle = G(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ip_0x}, \quad (2)$$

where  $G(x)$  is a broad Gaussian in real space centered at  $x_0 = 0$ ;  $p_0$  (assumed to be zero below) is the central momentum of the wavepacket along  $x$ . To better understand ZB, the dynamics can be analyzed in the momentum space. Hence we carry out the Fourier transformation of the wave function to get a narrow wavepacket in momentum representation (the required narrowness will be explained later),

$$\langle p_x|\psi\rangle = g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ix_0p_x}, \quad (3)$$

where  $g(p_x)$  is also a Gaussian as the Fourier transformation of  $G(x)$ .

With the initial state specified in Eq. (3), the internal state will evolve because it effectively experiences two “magnetic fields”: one along  $y$  of strength  $C_0$  and the other along  $x$  with strength  $Dp_x$  for the component  $p_x$  [see Eq. (1)]. The total effective magnetic field strength is  $\sqrt{C_0^2 + p_x^2 D^2}$  and the direction of the total magnetic field is characterized by an angle  $\arctan(Dp_x/C_0)$ . To obtain analytical results, we keep effects up to the first order of  $Dp_x/C_0$ , thus making approximations  $\arctan(p_x D/C_0) \approx p_x D/C_0$  and  $\sqrt{C_0^2 + p_x^2 D^2} \approx C_0$ . Physically, our approximation is to assume that for different  $p_x$  components, their effective Zeeman splitting is assumed to be almost the same, but with the internal state precessing around slightly different directions characterized by  $\arctan(Dp_x/C_0)$ . In particular, the analytical solution to the Dirac-like equation (1) can be directly obtained (for  $C = C_0$ ), i.e.,

$$\begin{aligned} |\psi(t)\rangle &= \cos\left(\frac{1}{2}C_0t\right) g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ix_0p_x} \\ &+ \sin\left(\frac{1}{2}C_0t\right) g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(x_0 + \frac{D}{C_0})p_x}. \end{aligned} \quad (4)$$

Here an extra  $D$ -dependent phase factor  $e^{[-i(D/C_0)p_x]}$  in the second term arises from the  $p_x$ -dependence of the spinor precession axis. The meaning of our early assumption of a narrow wavepacket in the momentum space is also clear: the main part of the wavepacket should ensure that  $|p_x D| \ll C_0$ . Considering effects due to  $(p_x D/C_0)^2$  or high orders will capture some insignificant deformation of the wavepacket and the damping of ZB, but as demonstrated by our full numerical results below, these high-order effects will not change the essence of the physics under discussion and can be neglected here.

Interestingly, the analytical solution in Eq. (4) can be interpreted as a superposition of two sub-wavepackets centered at  $x_0 = 0$  and at  $x = x_0 + D/C_0 = D/C_0$ . The

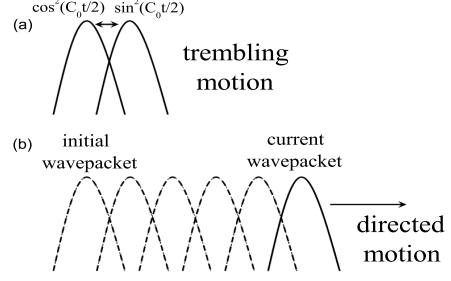


FIG. 1: Schematic plot of wavepacket motion satisfying a Dirac-like equation. In case (a), a Dirac-like equation is not modulated. Cyclic population transfer between two sub-wavepackets gives rise to ZB oscillation. In case (b), a Dirac-like equation is periodically modulated, with the modulation frequency on-resonance with the ZB frequency. Due to the on-resonance modulation, sub-wavepackets successively reemerge in different positions, yielding directed motion.

occupation probabilities of the two sub-wavepackets are given by  $\cos^2(C_0 t/2)$  and  $\sin^2(C_0 t/2)$ . As time evolves, these two occupation probabilities oscillate, thus giving rise to a time-dependence of the average position denoted  $\langle x \rangle$ . The jittering motion is between  $x = 0$  and  $x = D/C_0$ , with the angular frequency  $C_0$ . Specifically,  $\langle x \rangle$  is given by

$$\begin{aligned} \langle x \rangle &= \cos^2(C_0 t/2)x_0 + \sin^2(C_0 t/2)(x_0 + D/C_0) \\ &= \frac{D}{2C_0}[1 - \cos(C_0 t)], \end{aligned} \quad (5)$$

which is nothing but a ZB phenomenon, with the ZB frequency  $C_0$ , the ZB period  $T_{ZB} = 2\pi/C_0$ , and the “equilibrium point” of the oscillation at  $x = D/(2C_0)$ . In particular, when  $t = \pm\pi/C_0$ , the sub-wavepacket located at  $x = 0$  disappears and the other one located at  $x = D/C_0$  reaches its maximal population. Figure 1(a) schematically depicts this ZB mechanism.

Given this interpretation of ZB in terms of the population cycling between two sub-wavepackets, we next investigate what if the sign of the  $\sigma_y$  term is changed at  $t = \pi/C_0 = T_{ZB}/2$ , with the modified Hamiltonian given by

$$H = \frac{1}{2}(-C_0\sigma_y + Dp_x\sigma_x). \quad (6)$$

At that instant, the state of the system is given by

$$\langle p_x|\psi(t = \pi/C_0)\rangle = g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(\frac{D}{C_0})p_x}. \quad (7)$$

Adopting the same approximations as introduced above, we analytically obtain

$$\begin{aligned} |\psi(t)\rangle &= \cos[C_0(t - \pi/C_0)/2] g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(\frac{D}{C_0})p_x} \\ &+ \sin[C_0(t - \pi/C_0)/2] g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i(\frac{2D}{C_0})p_x} \end{aligned} \quad (8)$$

Remarkably, Eq. (8) represents a superposition of a sub-wavepacket located at  $x = D/C_0$  and a new one at  $x = 2D/C_0$ . The new equilibrium position of ZB is hence shifted to  $x = 3D/(2C_0)$ . Loosely speaking, this indicates that if a sign change of the  $C\sigma_y$  term occurs at the right moment, then the  $D$ -dependent extra phase factor [see Eqs. (4) and (8)] in the time-evolving state makes an opposite contribution and consequently a sub-wavepacket at a new location emerges. From this new solution, it is also observed that at  $t = 2\pi/C_0 = T_{\text{ZB}}$ , the occupation probability of the sub-wavepacket centered at  $x = D/C_0$  will totally vanish and that of the other centered at  $x = 2D/C_0$  will become unity. The average position of the system moves from  $x = D/C_0$  at  $t = T_{\text{ZB}}/2$  to  $x = 2D/C_0$  at  $t = T_{\text{ZB}}$ .

Repeating this strategy, i.e., abruptly changing the sign of  $C$  after every interval of  $T_{\text{ZB}}/2$ , a directed transport emerges from a modulated ZB. That is, if we let

$$C = \begin{cases} C_0, & \eta \in [0, \frac{1}{2}) \\ -C_0, & \eta \in [\frac{1}{2}, 1) \end{cases}, \quad (9)$$

where  $\eta \equiv \text{mod}(\frac{t}{T_{\text{ZB}}}, 1)$ , then ZB physics no longer induces oscillations. Instead, ZB is forced to become directed motion, as schematically illustrated in Fig. 1(b). Certainly, modulating the  $C\sigma_y$  term in a slightly different manner may also result in directed transport in the  $-x$  direction. We will not repeat the similar theoretical analysis. In addition, as shown below, tailored modulation of  $C$  can lead to more complicated control over the wavepacket dynamics.

### III. NUMERICAL SIMULATION

In this section, we verify our analytical results by numerical simulations without any approximation. Two representative modulation schemes are considered. In scheme (i), we introduce the modulation of  $C$  as described by Eq. (9) first and then stop the modulation for  $T_{\text{stop}} = NT_{\text{ZB}}$  ( $N$  integer), followed by the same modulation again [see Fig. 2(a)]. In scheme (ii), everything is the same except that  $2T_{\text{stop}}/T_{\text{ZB}}$  is an odd integer [see Fig. 2(b)].

For scheme (i), Fig. 2(c) shows that the wavepacket first moves in the  $x$  direction, then undergoes ZB once the modulation is stopped, and finally the directed motion continues after the modulation is switched on again. The wavepacket profile in real space at different times is plotted in Fig. 2(e). For scheme (ii), Fig. 2(d) shows that the wavepacket first moves in the same way as in the first scheme. However, once the modulation is re-launched, the wavepacket transport is reversed. So this behavior may be regarded as a type of “super-ZB” oscillations (analogous to super-Bloch oscillations [13]). As also shown in Figs. 2(e) and 2(f), in both schemes the distortion of the wavepacket profile for the shown time

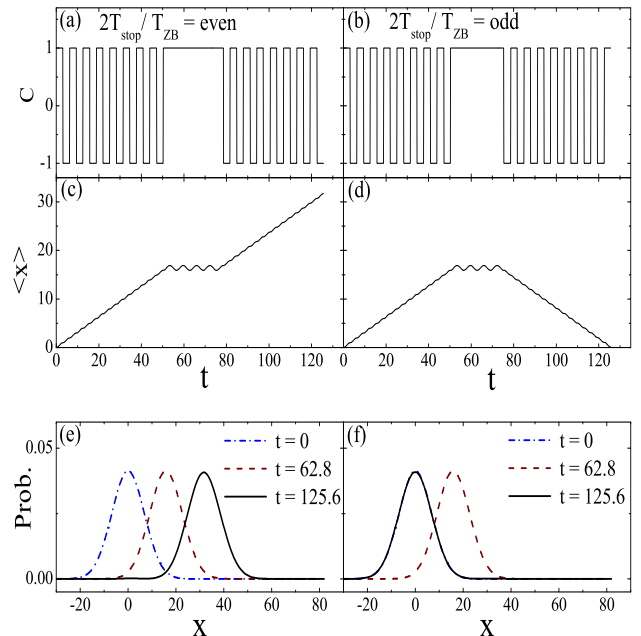


FIG. 2: (color online) Numerical results for a modulated Dirac-like equation. Panels (a) and (b) illustrate two schemes of modulation of the sign of system parameter  $C$ . When modulation is on, the sign of  $C$  is reversed after each time interval of  $T_{\text{ZB}}/2$  [see Eq. (9)]. The modulation off-period  $T_{\text{stop}}$  is an integer multiple of  $T_{\text{ZB}}$  in (a) and a half-integer multiple of  $T_{\text{ZB}}$  in (b). Panels (c) and (d) depict the time-dependence of the expectation value of  $\langle x \rangle$ , for modulation schemes in (a) and (b), respectively. It is seen that in (c), the wavepacket continues its directed motion after the modulation is switched on again; but in (d), the wavepacket eventually reverses its directed motion. Panels (e) and (f) show the wavepacket profile, i.e., probability density distribution in  $x$ , at three different times, for modulation schemes in (a) and (b), respectively. It is seen that wavepacket distortion is hardly visible. Note that in (f), the wavepacket at  $t = 125.6$  is on top of that at  $t = 0$ . All plotted quantities and system parameters are in dimensionless units, with  $C_0 = D = 1$ .

scale is invisible to our naked eyes. The numerical results here confirm our theoretical analysis, demonstrating that ZB can be indeed converted to directed wavepacket motion in a well-controlled fashion.

In real experiments, it should be more straightforward to introduce a smooth modulation to system parameters. Consider then a sinusoidal modulation of  $C$ ,  $C = C_0 \sin(\omega t)$ . This case is somewhat complicated because the magnitude of  $C$ , and hence the ZB frequency, is also time-dependent. To resolve this technical issue, we define a time-averaged effective ZB angular frequency over one half period of modulation, i.e.,

$$\omega_{\text{ZB}}^{\text{eff}} \equiv \frac{\omega}{\pi} \int_{t=0}^{\pi/\omega} C_0 \sin(\omega t) dt = \frac{2C_0}{\pi}. \quad (10)$$

In the light of our previous interpretation of ZB oscillation, it is expected that after the time interval  $\pi/\omega_{\text{ZB}}^{\text{eff}}$ , one of the two sub-wavepackets has vanished and the

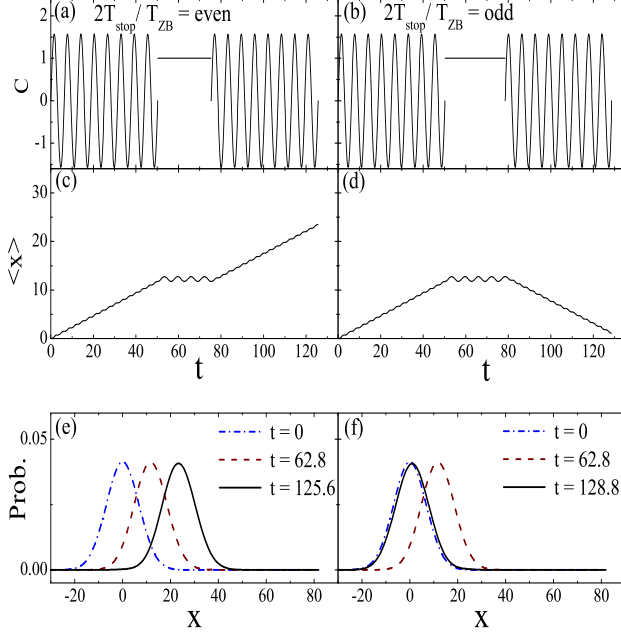


FIG. 3: (color online) Same as in Fig. 2, except for a sinusoidal modulation of  $C$ , i.e.,  $C = C_0 \sin(\omega t)$ , with  $\omega = 1$ ,  $C_0 = \pi/2$ , and  $D = 1$ . The pattern shown in panels (c)-(f) here are much similar to those seen in Fig. 2(c)-2(f), but the distance traveled by the wavepackets is different.

other has an occupation probability of unity. In accord with our theoretical insights above, the sign of  $C$  must be changing at this moment (from  $0^-$  to  $0^+$  or vice versa) in order to convert ZB oscillation to directed motion. This suggests the following modified resonance condition:  $\omega = \omega_{\text{ZB}}^{\text{eff}} = \frac{2C_0}{\pi}$ . It should be also noted that for a sinusoidal modulation, the magnitude of  $C$  can be very small during certain time windows and hence our early treatment in terms of the first-order expansion of  $D/C$  can be somewhat problematic. Nevertheless, as seen in Fig. 3, the numerical results confirm that  $\omega = \omega_{\text{ZB}}^{\text{eff}}$  is the

required resonance condition under a sinusoidal modulation so that ZB oscillation is again converted to directed motion. This also implies that the controlled dynamics is mainly contributed by the time segments during which  $C$  is appreciably nonzero. As seen in Fig. 3, for either even or odd  $2T_{\text{stop}}/T_{\text{ZB}}$  ( $T_{\text{ZB}}$  is now determined by  $\omega_{\text{ZB}}^{\text{eff}}$ ), the pattern of the results presented in Fig. 3(c)-(f) is essentially the same as those seen in Fig. 2(c)-(f), with again little wavepacket distortion. Note also that here the wavepacket transport distance per modulation cycle is less than that achieved by a mere discontinuous sign modulation of  $C$ .

#### IV. DISCUSSION

The Dirac-like equation in Eq. (1) can be realized in cold-atom systems or in trapped-ion systems. According to the concrete experimental realization in Ref. [11], the  $C\sigma_y$  term can be realized through two on-resonance laser fields, with the system parameter  $C$  connected with the phase and the strength of the laser fields. Modulation of  $C$  can hence be realized by modulating the laser phase, via a laser phase modulator or even an oscillating mirror [8]. The ZB frequency in Ref. [11] can be several tens of kHz, and the required phase modulation frequency should match this ZB frequency. Again using the experimental parameters in Ref. [11], it can be estimated that for a ZB frequency of 40 kHz, a wavepacket can be transported by about  $10^{-8}\text{m}$  per modulation cycle.

In summary, using a simple Dirac-like equation that is already experimentally realized, we show that ZB can be converted to directed motion via a periodic driving of the Dirac-like equation, if the driving is on resonance with the ZB oscillation. We hope that our results can motivate further studies of quantum control in driven Dirac-like equations.

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